Volume of random real algebraic submanifolds

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Random geometry

(M, g) compact Riemannian manifold of dimension n (without boundary).
We choose a codimension r submanifold of M "randomly".

Question

What can we say of the topology or the geometry of the submanifold?
(volume, Euler characteristic, number of connected components, ...)

We look for a statistical answer (mean, variance, distribution, ...) or an almost sure behavior.

Roots of real polynomials

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Question

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Theorem (Kac, 1943)

Let $P = \sum_{i=0}^{d} a_i X^i$, *where* $a_0, \ldots, a_d$ *are i.i.d. standard Gaussian variables and let* $Z_d = P^{-1}(0)$, *then*

$$\mathbb{E}[\mathrm{card}\,(Z_d)] \sim \frac{2}{\pi}\log(d).$$

Higher dimensions

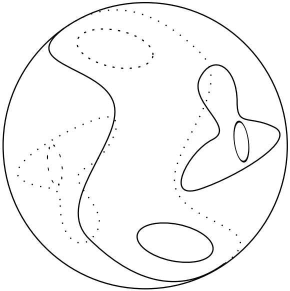
Notations

Let $\alpha = (\alpha_0, \dots, \alpha_n) \in \mathbb{N}^{n+1}$, we set:

- $|\alpha| = \alpha_0 + \cdots + \alpha_n$,
- $X^\alpha = X_0^{\alpha_0} \cdots X_n^{\alpha_n}$,
- $\alpha! = \alpha_0! \cdots \alpha_n!$,
- if $|\alpha| = d$, $\binom{d}{\alpha} = \frac{d!}{\alpha!}$.

P homogeneous polynomial in $\mathbb{R}_d^{\hom}[X_0, \dots, X_n]$: $P = \sum_{|\alpha|=d} a_\alpha X^\alpha$.

$P^{-1}(0) \subset \mathbb{R}^{n+1}$ is a cone.
We consider $Z_P = P^{-1}(0) \cap \mathbb{S}^n$.



What is a manifold?

Definition

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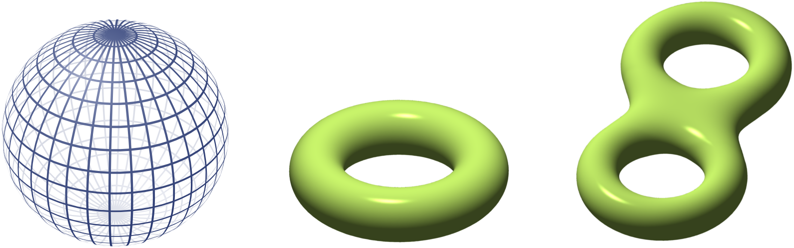
It generalizes the idea of a non-singular curve or surface (no double points, no cusps, etc.).



Source: `en.wikipedia.org`

Main point

We can extend the calculus to maps between manifolds.

What is a submanifold?

Let M be a manifold of dimension n and $r \in \{1, \ldots, n\}$.

Definition

A codimension r submanifold of M is $Z_f \subset M$ such that $Z_f = f^{-1}(0)$, where:

- $f : M \to \mathbb{R}^r$ is smooth,
- for all x such that $f(x) = 0$, $d_x f$ is surjective.

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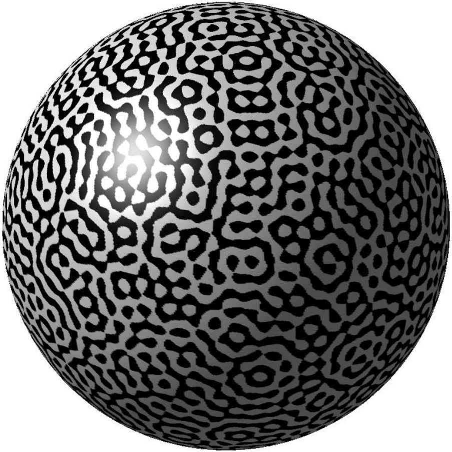
Main point

Z_f is a manifold of dimension $n - r$.

A random curve on the sphere



Picture by Alex Barnett (Dartmouth).

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On a Riemannian manifold (M, g), there are:

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- a natural volume measure $|\mathrm{d}V_M|$.

If Z_f is a codimension r submanifold of M, the restriction of g is a Riemannian metric on Z_f.
We denote by $|\mathrm{d}V_f|$ the associated ($(n-r)$-dimensional) volume measure.

Gaussian variables

$(V, \langle \cdot \,, \cdot \rangle)$ Euclidean space of dimension N,
Λ self-adjoint and positive definite.

Definition

A random vector X in V is a centered Gaussian of variance Λ if its distribution admits the density:

$$\frac{1}{(2\pi)^{\frac{N}{2}}\sqrt{\det(\Lambda)}} \exp\left(-\frac{1}{2}\left\langle \Lambda^{-1}x \,, x\right\rangle\right)$$

with respect to the Lebesgue measure. This is denoted by $X \sim \mathcal{N}(\Lambda)$.

We say that $X \sim \mathcal{N}(\mathrm{Id})$ is a standard Gaussian.
In any orthonormal basis $(e_1, \ldots, e_N)$ we have $X = \sum a_i e_i$, where $a_i \sim \mathcal{N}(1)$ are i.i.d. real random variables.

Some properties of Gaussian variables

- Two jointly Gaussian vectors are independent iff they are uncorrelated.

- If $X \sim \mathcal{N}(\Lambda)$ in V and $L : V \to V'$ is linear, then $L(X) \sim \mathcal{N}(L\Lambda L^*)$.

- If (X, Y) is a centered Gaussian vector with variance $\left(\begin{smallmatrix} A & B^{\mathrm{t}} \\ B & C \end{smallmatrix}\right)$, then the distribution of Y given that $X = 0$ is also a centered Gaussian vector, and its variance is:
$$C - BA^{-1}B^{\mathrm{t}}.$$

Kostlan–Shub–Smale polynomials

We consider a random Kostlan-distributed $P \in \mathbb{R}_d^{\text{hom}}[X_0, \dots, X_n]$. That is:

$$P = \sqrt{\frac{(d+n)!}{\pi^n d!}} \sum_{|\alpha|=d} a_\alpha \sqrt{\binom{d}{\alpha}} X^\alpha,$$

where $(a_\alpha)_{|\alpha|=d}$ are i.i.d. $\mathcal{N}(1)$ real variables.

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Remark

$P \sim \mathcal{N}(\text{Id})$ in $\mathbb{R}_d^{\text{hom}}[X_0, \ldots, X_n]$ for the following L^2 scalar product:

$$\langle P \, , Q \rangle = \frac{1}{2\pi} \int_{\{z \in \mathbb{C}^{n+1} | \|z\|=1\}} P(z) \overline{Q(z)} \, \mathrm{d}\theta(z).$$

Kostlan's distribution is invariant under the action of $O_{n+1}(\mathbb{R})$ by:

$$(O \cdot P)(x) = P(O^{-1} x).$$

Kostlan–Shub–Smale polynomials

Let $d, n \in \mathbb{N}^*$ and $r \in \{1, \ldots, n\}$,
$P_1, \ldots, P_r$ i.i.d. Kostlan-distributed polynomials in $\mathbb{R}_d^{\text{hom}}[X_0, \ldots, X_n]$.
We set $Z_d = Z_{P_1} \cap \cdots \cap Z_{P_r} \subset \mathbb{S}^n$.

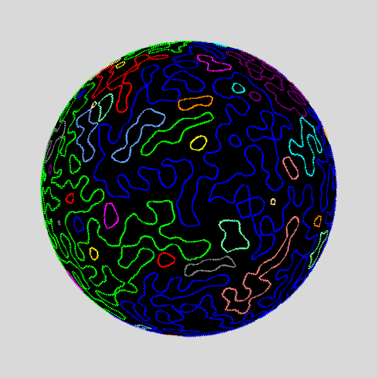
Lemma

Z_d is almost surely a codimension r submanifold of $\mathbb{S}^n$.

Theorem (Kostlan, 1993)

For all n, r and d, we have: $\mathbb{E}[\mathrm{Vol}\,(Z_d)] = d^{\frac{r}{2}}\,\mathrm{Vol}\,(\mathbb{S}^{n-r})$.

Kostlan–Shub–Smale polynomials



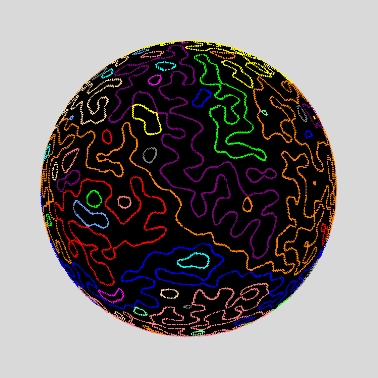


Figure: Degree 56 random curves in $\mathbb{S}^2$, Kostlan's model.

Pictures by Maria Nastasescu (Caltech).

Real algebraic framework

M real algebraic manifold of dimension n (for example $M = Z_P \subset \mathbb{S}^{n+1}$), with a natural Riemannian metric.

$(P_1, \ldots, P_r)$ replaced by a standard Gaussian section s of $\mathcal{E} \otimes \mathcal{L}^d$, a rank r real holomorphic Hermitian vector bundle, with $\mathcal{L}$ ample line bundle.
In this setting, $Z_d = s_d^{-1}(0)$.

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Lemma

If d is large enough, then Z_d is a.s. a codimension r submanifold of M.

Expected volume

$|\mathrm{d}V_M|$ Riemannian measure on M, $|\mathrm{d}V_d|$ Riemannian measure on Z_d.

We see Z_d as a measure on M: $\forall \phi \in \mathcal{C}^0(M), \langle Z_d \, , \phi \rangle = \displaystyle\int_{Z_d} \phi \, |\mathrm{d}V_d|$.

Theorem (L., 2014)

For all $\phi \in \mathcal{C}^0(M)$,

$$\mathbb{E}[\langle Z_d \, , \phi \rangle] = d^{\frac{r}{2}} \left(\int_M \phi \, |\mathrm{d}V_M| \right) \frac{\mathrm{Vol}\,(\mathbb{S}^{n-r})}{\mathrm{Vol}\,(\mathbb{S}^n)} + \|\phi\|_{\mathcal{C}^0} \, O\Big(d^{\frac{r}{2}-1}\Big) \, ,$$

where the error term is independent of ϕ.

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where the error term is independent of ϕ.

Corollary

As Radon measures, we have: $d^{-\frac{r}{2}}\mathbb{E}[Z_d] \xrightarrow[d\to+\infty]{} \frac{\mathrm{Vol}\left(\mathbb{S}^{n-r}\right)}{\mathrm{Vol}\left(\mathbb{S}^{n}\right)}\,|\mathrm{d}V_M|$.

Variance of the volume

Theorem (L., 2016)

If $1 \leqslant r < n$, then for all $\phi \in \mathcal{C}^0(M)$,

$$\mathrm{Var}(\langle Z_d \, , \phi \rangle) = d^{r-\frac{n}{2}} \left(\int_M \phi^2 \, |\mathrm{d}V_M| \right) \mathcal{I}_{n,r} + o\Big(d^{r-\frac{n}{2}}\Big) \, ,$$

where $\mathcal{I}_{n,r}$ is explicit and $0 \leqslant \mathcal{I}_{n,r} < +\infty$.

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where $\mathcal{I}_{n,r}$ *is explicit and* $0 \leqslant \mathcal{I}_{n,r} < +\infty$*.*

Corollary

$$\mathrm{Var}(\mathrm{Vol}\,(Z_d)) = d^{r-\frac{n}{2}}\,\mathrm{Vol}\,(M)\,\mathcal{I}_{n,r} + o\!\left(d^{r-\frac{n}{2}}\right).$$

The case of random points

For one Kostlan–Shub–Smale polynomial in $\mathbb{S}^1$ ($n = r = 1$).

Theorem (Kostlan, 1993)

$$\mathbb{E}[\mathrm{card}(Z_d)] = 2\sqrt{d}.$$

Theorem (Dalmao, 2015)

There exists $\sigma^2 > 0$ explicit such that:

$$\mathrm{Var}(\mathrm{card}(Z_d)) \sim \sigma^2\sqrt{d}.$$

Moreover,

$$\frac{\mathrm{card}(Z_d) - 2\sqrt{d}}{\sigma d^{\frac{1}{4}}} \xrightarrow[d\to+\infty]{\mathcal{D}} \mathcal{N}(1).$$

Concentration around the mean

Corollary

If $1 \leqslant r < n$, then for all $\phi \in \mathcal{C}^0(M)$,

$$\mathbb{P}\left(\left|\frac{\langle Z_d\,,\phi\rangle - \mathbb{E}[\langle Z_d\,,\phi\rangle]}{d^{\frac{r}{2}}}\right| \geqslant \frac{1}{d^{\frac{r}{4}}}\right) = O\Big(d^{\frac{r-n}{2}}\Big).$$

By Markov's inequality,

$$\begin{aligned}\mathbb{P}\left(\left|\frac{\langle Z_d\,,\phi\rangle - \mathbb{E}[\langle Z_d\,,\phi\rangle]}{d^{\frac{r}{2}}}\right| \geqslant \frac{1}{d^{\frac{r}{4}}}\right) &= \mathbb{P}\left(\Big|\langle Z_d\,,\phi\rangle - \mathbb{E}[\langle Z_d\,,\phi\rangle]\Big| \geqslant d^{\frac{r}{4}}\right) \\ &\leqslant d^{-\frac{r}{2}}\,\mathrm{Var}(\langle Z_d\,,\phi\rangle) \\ &= O\Big(d^{\frac{r-n}{2}}\Big).\end{aligned}$$

Equidistribution in probability

Corollary

If $1 \leqslant r < n$*, for every open set* $U \subset M$*, we have:*

$$\mathbb{P}\left(Z_d \cap U = \emptyset\right) = O\left(d^{-\frac{n}{2}}\right).$$

Equidistribution in probability

Corollary

If $1 \leqslant r < n$, for every open set $U \subset M$, we have:

$$\mathbb{P}\left(Z_d \cap U = \emptyset\right) = O\left(d^{-\frac{n}{2}}\right).$$

Let $\phi_U \in \mathcal{C}^0(M)$ that vanishes outside U and is positive on U.

Let $\varepsilon > 0$ be such that, for every d large enough,

$$\mathbb{E}[\langle Z_d \, , \phi_U\rangle] - d^{\frac{r}{2}}\varepsilon > 0.$$

Equidistribution in probability

$$\begin{aligned}
\mathbb{P}\left(Z_d \cap U = \emptyset\right) &= \mathbb{P}\left(\langle Z_d\,, \phi_U\rangle = 0\right) \\
&\leqslant \mathbb{P}\left(\langle Z_d\,, \phi_U\rangle < \mathbb{E}[\langle Z_d\,, \phi_U\rangle] - d^{\frac{r}{2}}\varepsilon\right) \\
&\leqslant \mathbb{P}\left(\left|\langle Z_d\,, \phi_U\rangle - \mathbb{E}[\langle Z_d\,, \phi_U\rangle]\right| > d^{\frac{r}{2}}\varepsilon\right) \\
&\leqslant \frac{1}{d^r\varepsilon^2}\operatorname{Var}(\langle Z_d\,, \phi_U\rangle) \\
&= O\!\left(d^{-\frac{n}{2}}\right).
\end{aligned}$$

Universality of the zero set

Theorem (Gayet–Welschinger, 2013)

Let $\Sigma \subset \mathbb{R}^n$ be a codimension r compact submanifold without boundary and $R > 0$.
Then, there exists $C_{\Sigma,R} \geqslant 0$ such that, for all d large enough, for all $x \in M$,

$$\mathbb{P}\left(Z_d \cap B\left(x, \frac{R}{\sqrt{d}}\right) \supset \Sigma' \text{ s.t. } \left(B\left(x, \frac{R}{\sqrt{d}}\right), \Sigma'\right) \simeq (\mathbb{R}^n, \Sigma)\right) \geqslant C_{\Sigma,R}.$$

Moreover, $C_{\Sigma,R} > 0$ for R large enough.

The correlation function

A Kostlan polynomial $P \in \mathbb{R}_d^{\mathrm{hom}}[X_0, \dots, X_n]$ defines a centered Gaussian process $(P(x))_{x \in \mathbb{S}^n}$.

This process is characterized by its correlation function:

$$e_d : (x, y) \mapsto \mathbb{E}[P(x)P(y)] .$$

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$$\mathbb{E}[P(x)P(y)] = \frac{(d+n)!}{\pi^n d!} \sum_{|\alpha|=d} \binom{d}{\alpha} x^\alpha y^\alpha = \frac{(d+n)!}{\pi^n d!} \left(\langle x \,, y \rangle\right)^d$$

$$= \frac{(d+n)!}{\pi^n d!} \cos\left(D(x,y)\right)^d \,.$$

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Remark

By taking partial differentials, $\frac{\partial e_d}{\partial x_i}(x,y) = \mathbb{E}\left[\frac{\partial P}{\partial x_i}(x) P(y)\right]$.

Scaling limit of the Bergman kernel

In the general real algebraic framework, the correlation function e_d is the Bergman kernel of $\mathcal{E} \otimes \mathcal{L}^d$.

There is a universal scaling limit for e_d (Ma–Marinescu, 2007) :

$$e_d(x, y) \simeq \frac{d^n}{\pi^n} \exp\left(-\frac{d}{2}\|x - y\|^2\right),$$

whenever $D(x, y) \leqslant K\frac{\log d}{\sqrt{d}}$.

Theorem (Ma–Marinescu, 2015)

There exists $C > 0$ such that, for all $k \in \mathbb{N}$,

$$\|e_d(x, y)\|_{\mathcal{C}^k} = O\Big(d^{n+\frac{k}{2}} \exp\left(-C\sqrt{d}D(x, y)\right)\Big),$$

uniformly in (x, y).

A heuristic for the mean volume

The Bergman kernel shows a characteristic scale $\frac{1}{\sqrt{d}}$.

We cut M into boxes of size $\frac{1}{\sqrt{d}}$:

$$\simeq \mathrm{Vol}\,(M)\, d^{\frac{n}{2}} \text{ boxes.}$$

The boxes are independent, same distribution of Z_d in each box.

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Components of size $\frac{1}{\sqrt{d}}$, each one has a volume of order $\left(\frac{1}{\sqrt{d}}\right)^{n-r}$.
Finally, $\mathrm{Vol}\,(Z_d)$ is of order $\mathrm{Vol}\,(M)\, d^{\frac{r}{2}}$.

Kac–Rice formula

In the case of hypersurfaces ($r = 1$).

$P \in \mathbb{R}_d^{\text{hom}}[X_0, \ldots, X_n]$ Kostlan-distributed, $Z_d = P^{-1}(0) \cap \mathbb{S}^n$.

Kac–Rice formula

For every ϕ,

$$\mathbb{E}\left[\int_{Z_d} \phi \, |\mathrm{d}V_d|\right] = \frac{1}{\sqrt{2\pi}} \int_{x \in \mathbb{S}^n} \phi(x) \frac{\mathbb{E}\left[\|d_x P\| \,\middle|\, P(x) = 0\right]}{\sqrt{e_d(x,x)}}.$$

$x \mapsto e_d(x,x)$ does not vanish (i.e. for all $x \in M$, $P(x)$ is non-degenerate).
Similar formula in the general case.

Asymptotic of the expectation

$$\mathbb{E}[\langle Z_d\,,\phi\rangle] = \frac{1}{\sqrt{2\pi}}\int_{x\in\mathbb{S}^n}\phi(x)\frac{\mathbb{E}\left[\|d_xP\|\,\middle|\,P(x)=0\right]}{\sqrt{e_d(x,x)}}.$$

We have $e_d(x,x)\sim\frac{d^n}{\pi^n}$. We need to estimate $\mathbb{E}\left[\|d_xP\|\,\middle|\,P(x)=0\right]$.

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$(P(x),d_xP)$ is a centered Gaussian of variance:

$$\Lambda = \begin{pmatrix} e_d(x,x) & \partial_{y_1}e_d(x,x) & \cdots & \partial_{y_n}e_d(x,x) \\ \partial_{x_1}e_d(x,x) & \partial_{x_1}\partial_{y_1}e_d(x,x) & \cdots & \partial_{x_1}\partial_{y_n}e_d(x,x) \\ \vdots & \vdots & \ddots & \vdots \\ \partial_{x_n}e_d(x,x) & \partial_{x_n}\partial_{y_1}e_d(x,x) & \cdots & \partial_{x_n}\partial_{y_n}e_d(x,x) \end{pmatrix}.$$

The conditional distribution of d_xP is a centered Gaussian.
We can compute its variance from Λ.

Asymptotic of the expectation

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We can compute its variance from Λ.
We conclude by using the estimates of Ma and Marinescu for e_d.

Asymptotic of the variance

$$\mathrm{Var}(\langle Z_d\,,\phi\rangle) = \mathbb{E}\left[\langle Z_d\,,\phi\rangle^2\right] - \mathbb{E}[\langle Z_d\,,\phi\rangle]^2\,.$$

By the Kac–Rice formula, $\mathbb{E}[\langle Z_d\,,\phi\rangle]^2$ equals:

$$\frac{1}{2\pi}\int_{x,y\in\mathbb{S}^n}\phi(x)\phi(y)\frac{\mathbb{E}\left[\|d_xP\|\,\middle|\,P(x)=0\right]}{\sqrt{e_d(x,x)}}\frac{\mathbb{E}\left[\|d_yP\|\,\middle|\,P(y)=0\right]}{\sqrt{e_d(y,y)}}.$$

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Besides,

$$\mathbb{E}\left[\langle Z_d\,,\phi\rangle^2\right] = \mathbb{E}\left[\int_{x,y\in Z_d}\phi(x)\phi(y)\,|\mathrm{d}V_d|^2\right].$$

Kac–Rice formula 2

For d large enough, we have $\det \begin{pmatrix} e_d(x,x) & e_d(x,y) \\ e_d(y,x) & e_d(y,y) \end{pmatrix} \neq 0$, whenever $x \neq y$ (i.e. $(P(x), P(y))$ is non-degenerate).

Kac–Rice formula

$$\mathbb{E}\left[\int_{x,y\in Z_d} \phi(x)\phi(y)\,|\mathrm{d}V_d|^2\right] =$$

$$\frac{1}{2\pi}\int_{x,y\in\mathbb{S}^n} \phi(x)\phi(y)\frac{\mathbb{E}\left[\|d_xP\|\,\|d_yP\|\,\middle|\,P(x)=0=P(y)\right]}{\sqrt{e_d(x,x)e_d(y,y)-e_d(x,y)^2}}.$$

Asymptotic of the variance

Finally

$$\mathrm{Var}(\langle Z_d\,,\phi\rangle) = \frac{1}{2\pi}\int_{x,y\in\mathbb{S}^n} \phi(x)\phi(y)\mathcal{D}_d(x,y),$$

where

$$\mathcal{D}_d(x,y) = \frac{\mathbb{E}\left[\|d_xP\|\,\|d_yP\| \,\middle|\, P(x)=0=P(y)\right]}{\sqrt{e_d(x,x)e_d(y,y)-e_d(x,y)^2}} - \frac{\mathbb{E}\left[\|d_xP\| \,\middle|\, P(x)=0\right]}{\sqrt{e_d(x,x)}}\frac{\mathbb{E}\left[\|d_yP\| \,\middle|\, P(y)=0\right]}{\sqrt{e_d(y,y)}}.$$

Asymptotic of the variance

When $D(x,y) \geqslant K\frac{\log d}{\sqrt{d}}$, we prove that $\mathcal{D}_d(x,y)$ is $O\left(d^{r-\frac{n}{2}-1}\right)$.
Moreover, $\frac{1}{d^r}\mathcal{D}_d\left(x, x+\frac{z}{\sqrt{d}}\right) \xrightarrow[d\to+\infty]{} \mathcal{D}(z)$.

$$\begin{aligned}\int_{x,y\in\mathbb{S}^n} \phi(x)\phi(y)\mathcal{D}_d(x,y) &\simeq \int_{x\in\mathbb{S}^n}\int_{y\in B(x,\frac{\log d}{\sqrt{d}})} \phi(x)\phi(y)\mathcal{D}_d(x,y)\\ &\simeq d^{-\frac{n}{2}}\int_{x\in\mathbb{S}^n}\int_{z\in B(0,\log d)} \phi(x)\phi\left(x+\frac{z}{\sqrt{d}}\right)\mathcal{D}_d\left(x,x+\frac{z}{\sqrt{d}}\right)\\ &\simeq d^{r-\frac{n}{2}}\left(\int_{x\in\mathbb{S}^n}\phi(x)^2\right)\left(\int_{\mathbb{R}^n}\mathcal{D}(z)\right).\end{aligned}$$

Almost sure equidistribution

We consider a random sequence of polynomials of increasing degree

$$(P_d)_{d\in\mathbb{N}^*} \in \prod_{d\in\mathbb{N}^*} \mathbb{R}_d^{\mathrm{hom}}[X_0,\dots,X_n],$$

distributed as $\mathrm{d}\nu$, product measure of the Kostlan distributions.

Corollary

If $n \geqslant 3$, then $\mathrm{d}\nu$-almost surely we have:

$$\forall \phi \in \mathcal{C}^0(\mathbb{S}^n), \qquad \frac{1}{\sqrt{d}}\left\langle Z_{P_d}\,,\phi\right\rangle \xrightarrow[d\to+\infty]{} \frac{\mathrm{Vol}\left(\mathbb{S}^{n-1}\right)}{\mathrm{Vol}\left(\mathbb{S}^{n}\right)}\int_{\mathbb{S}^n}\phi.$$

Almost sure equidistribution

Let $\phi \in \mathcal{C}^0(\mathbb{S}^n)$, we have:

$$\mathbb{E}\left[\sum_{d \geqslant 1} \left(\frac{1}{\sqrt{d}} \left(\langle Z_{P_d}, \phi \rangle - \mathbb{E}[\langle Z_d, \phi \rangle] \right) \right)^2 \right] = \sum_{d \geqslant 1} \frac{1}{d} \operatorname{Var}(\langle Z_d, \phi \rangle) < +\infty,$$

since $\operatorname{Var}(\langle Z_d, \phi \rangle) = O\left(d^{1-\frac{3}{2}}\right)$. Then d$\nu$-a.s.

$$\sum_{d \geqslant 1} \left(\frac{1}{\sqrt{d}} \left(\langle Z_{P_d}, \phi \rangle - \mathbb{E}[\langle Z_d, \phi \rangle] \right) \right)^2 < +\infty,$$

and

$$\frac{1}{\sqrt{d}} \langle Z_{P_d}, \phi \rangle \xrightarrow[d \to +\infty]{d\nu-\text{p.s.}} \frac{\operatorname{Vol}\left(\mathbb{S}^{n-1}\right)}{\operatorname{Vol}\left(\mathbb{S}^n\right)} \int_{\mathbb{S}^n} \phi.$$

We conclude by using the separability of $\mathcal{C}^0(\mathbb{S}^n)$.

The end

Thank you for your attention.